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Multifractal Analysis of Infinite Products

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We construct a family of measures called infinite products which generalize Gibbs measures in the one-dimensional lattice gas model. The multifractal properties of these measures are studied under some regularity conditions. In particular, if the τ-function is differentiable, we prove a formula which gives the Hausdorff dimension and packing dimension of the set of singularity points of a given order. Mathematical examples include Riesz products, g-measures, and G-measures.

KEY WORDS: Gibbs measure; Riesz product; pressure function; multifractal analysis.

1. INTRODUCTION

Consider a system with infinite sites represented by the set of integers $\mathbf{Z}$ and with finite states represented by $\{1, 2,..., q\}(q \geqslant 2)$. Then a configuration of the system is a sequence $x=(x_n)$ in $D=\{1, 2,..., q\}^{\mathbf{Z}}$. For such a configuration, we define the energy contribution due to x_k by

$$\psi_k(x) = \Psi_k^{(1)}(x_k) + \frac{1}{2} \sum_{j \neq k} \Psi_k^{(2)}(j, k; x_j, x_k)$$

where $\Psi_k^{(1)}(x_k)$ means the energy contribution of the occurrence x_k at the site k and $\Psi_k^{(2)}(j, k; x_j, x_k)$ means the energy contribution of x_j toward x_k. We distinguish two special cases. If $\Psi_k^{(2)}(j, k; x_j, x_k) = \Psi_j^{(2)}(k, j; x_k, x_j)$, we say the system is *symmetric*; we say the system is *symmetric and homogeneous* if, moreover, $\Psi_k^{(2)}(j, k; x_j, x_k) = \Psi^{(2)}(|j-k|; x_j, x_k)$ for some function $\Psi^{(2)}$ independent of k. Symmetric and homogeneous systems are studied in refs. 2, 8 and 28. We are interested here in symmetric systems,

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not necessarily homogeneous. If we look at the system restricted on $[-n, n] \cap \mathbf{Z}$, the total energy of $(x_{-n},..., x_0,..., x_n)$ is equal to

$$\sum_{k=-n}^{n} \psi_k(\sigma^k x)$$

where σ is the shift transformation on D.

Our objective is to study the limit behavior of this energy when $n \to \infty$. For convenience, we shall study the one-sided shift space $D^+ = \{1, 2,..., q\}^{\mathbf{N}}$. Then the energy becomes $\sum_{k=0}^{n} \psi_k(\sigma^k x)$. Instead of working with D^+, we shall actually work with the interval $[0, 1)$, or equivalently the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, because there is a natural correspondence between the two spaces. The limit of the energy is then described by the the so-called Gibbs measures, whose existence will be proved under some additional conditions on ψ_k. These conditions also allow us to prove the uniqueness of the Gibbs measure and the existence of pressure, to estimate the Hausdorff dimensions of the singularity point sets of the Gibbs measure through the large-deviation results. The required condition is proved to be satisfied in the almost-periodic cases (the constant case, i.e., $\psi_k \equiv \psi$, corresponds to the classical case[2, 28]) and in some random cases.

We shall work in a slightly more general case than that described above. Let $\{\lambda_n\}_{n \geqslant 0}$ be a lacunary sequence of positive integers satisfying the condition

$$\lambda_n \mid \lambda_{n+1} \quad (n \geqslant 0), \qquad 1 < \min_{n \geqslant 0} \frac{\lambda_{n+1}}{\lambda_n} \leqslant \max_{n \geqslant 0} \frac{\lambda_{n+1}}{\lambda_n} < \infty \tag{1}$$

(the above case corresponds to $\lambda_n = q^n$). Let $\{g_n\}_{n \geqslant 0}$ be a sequence of positive, periodic, continuous functions defined on the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ satisfying the conditions

$$\begin{gathered} \sup_{n \geqslant 0} \sum_{k=0}^{n} \Omega(\log g_k, \lambda_k/\lambda_n) < \infty \\ 0 < \inf_{n \geqslant 1, x \in \mathbf{T}} g_n(x) \leqslant \sup_{n \geqslant 1, x \in \mathbf{T}} g_n(x) < \infty \end{gathered} \tag{2}$$

where $\Omega(g, \delta) = \sup_{|x-y| \leqslant \delta} |g(x) - g(y)|$ denotes the modulus of continuity of a function g (the above case corresponds to $g_k = \exp \psi_k$). In this paper we study the weak limits of the sequence of measures $Z_n^{-1} P_n(t)\, dt (n \geqslant 1)$, where Z_n is a normalization defined by

$$Z_n = \int_0^1 P_n(t)\, dt \qquad \text{with} \quad P_n(t) = \prod_{k=0}^{n-1} g_n(\lambda_k t)$$

Actually, we shall see that all weak limits are strongly equivalent in that the relation $c^{-1}\nu(I) \leqslant \mu(I) \leqslant c\nu(I)$ holds for any two limit measures μ and ν, any interval I, and some constant $c > 0$ (see Proposition 2). We take one of these limit measures as a representation and denote it by μ. We call μ the *infinite product* defined by the generating functions $\{g_n\}$ based on the lacunary sequence $\{\lambda_n\}$. The study will be made under the supplementary condition that the limit

$$\varphi(\beta) = \lim_{n \to \infty} \frac{1}{\log \lambda_n} \log \int_0^1 [P_n(t)]^\beta \, dt \tag{3}$$

exists for every $\beta \in \mathbf{R}$. This function is called the *φ-function.* The existence of the limit (3) will be proved to be true in certain cases, for example, the case where $\lambda_n = q^n$ for some $q \geqslant 2$ and $\{g_n\}$ is periodic, or almost periodic, or stationary in the probabilistic sense.

Our aim is to give a multifractal analysis of such an infinite product measure. Multifractal analysis was motivated by models describing turbulence, diffusion-limited aggregation, percolation, etc.[21, 13, 14] Some rigorous results have already been obtained for some invariant measures in dynamical systems, [1, 6, 8, 9, 25, 29] for some self-similar measures generated by iterated function systems,[5, 18, 22] and for some quasiindependent measures.[4] Let us recall the precise problem. For a given measure μ on $\mathbf{T}$ and a given $\alpha \geqslant 0$, we are interested in the size of the set of singularity points of order α

$$E_\alpha = \left\{ x \in \mathbf{T} \colon \lim_{r \to 0} \frac{\log \mu(I_r(x))}{\log r} = \alpha \right\}$$

where $I_r(x)$ denotes the interval centered at x of length $2r$. Let $f(\alpha) = \dim E_\alpha$ denote the Hausdorff dimension of E_α. We could say that μ is *multifractal* if $f(\alpha) \neq 0$ for a continuum of α. By multifractal analysis we mean to provide a description of $f(\alpha)$ as precise as possible. It is more interesting of course to know the function $f(\alpha)$ in a precise manner. In refs. 13 and 14 a solution was suggested:

$$f(\alpha) = \inf_\beta (\alpha\beta - \tau(\beta))$$

where τ is the so-called *τ-function*, which is defined by

$$\tau(\beta) = \lim_{r \to 0} \frac{1}{\log r} \log \int_0^1 \mu(I_r(t))^{\beta - 1} \, d\mu(t)$$

if the limit exists. This formula is called the *singularity spectrum formula.* Our main result is that for an infinite product, we prove a weak form of the above singularity spectrum formula under conditions (1)–(3) and we prove the exact formula just mentioned if the function φ is differentiable (which is equivalent to saying that τ is differentiable). The same results are also proved for the packing dimension of E_α.

To some extent, infinite products generalize Riesz products,[35] g-measures,[15] and G-measures.[3] The results presented here can be applied to all these measures.

The paper is organized as follows. In Section 2 we prove a fundamental inequality which has its counterpart for symmetric and homogeneous systems in ref. 28 and the strong equivalence of two limit measures mentioned above. In Section 3 we study the φ-function, the τ-function, their conjugates, and their relation to large-deviation results. This is preparative to providing a limit theorem needed in Sections 4 and 5, where we study several different notions of dimensions, some of which are directly related to the τ-function, and we prove the singularity spectrum formula. We return to the existence of the limit (3) in Section 6, which is devoted to the deterministic almost-periodic case, and in Section 7, which is devoted to the random stationary case. Some examples are examined in Section 8.

2. FUNDAMENTAL INEQUALITY AND INFINITE PRODUCTS

We shall use the following notation throughout the text. For two quantities u and v, $u \approx v$ means "$c^{-1}v \leqslant u \leqslant cv$ for some $c > 0$"; for $t \in \mathbf{T}$, $I(t)$ means an interval containing t; $|I|$ means the length of an interval I. Recall that

$$P_n(t) = \prod_{k=0}^{n-1} g_k(\lambda_k t)$$

which is defined in the Introduction. Define, for $\beta \in \mathbf{R}$,

$$Z_n(\beta) = \int_0^1 (P_n(t))^\beta \, dt$$

Such a function is called a *partition function.*[28] More generally, we define for $0 \leqslant m \leqslant n$

$$P_{m,n} = \frac{P_n}{P_m}, \qquad Z_{m,n}(\beta) = \int_0^1 (P_{m,n}(t))^\beta \, dt$$

Lemma 1. Suppose that condition (2) is satisfied. There exists a constant $C>0$ such that

$$C^{-1}P_n(s)\leqslant P_n(t)\leqslant CP_n(s)$$

holds for all $n\geqslant 1$ and all t and s with $|t-s|\leqslant 1/\lambda_{n-1}$.

Proof. We have the identity

$$\frac{P_n(t)}{P_n(s)}=\exp\sum_{k=0}^{n-1}\left[\log g_k(\lambda_k t)-\log g_k(\lambda_k s)\right]$$

As $|t-s|\leqslant 1/\lambda_{n-1}$, the sum in the last identity is bounded by

$$\sum_{k=0}^{n-1}\Omega(\log g_k, \lambda_k/\lambda_{n-1})$$

By the condition (2), this sum is bounded by a constant D. We can then choose $C=e^D$. ▮

For $n\geqslant 1$, let Γ_n be the subgroup of $\mathbf{T}$ generated by $1/\lambda_{n-1}$ and $\mathscr{F}^n$ be the sub-Borel σ-field invariant under Γ_n. Let $\mathscr{E}$ denote the Lebesgue integral and $\mathscr{E}^n$ the conditional expectation with respect to $\mathscr{F}^n$. It is easy to see that for any $f\in L^1(\mathbf{T})$ we have

$$\mathscr{E}^n f(x)=\frac{1}{\lambda_{n-1}}\sum_{k=0}^{\lambda_{n-1}-1} f(x+k/\lambda_{n-1})$$

Lemma 2. Suppose that condition (2) is satisfied. For any $\beta\in\mathbf{R}$ we have a version of $\mathscr{E}^n P_n^\beta$ such that for all $n\geqslant 1$ and all $x\in\mathbf{T}$

$$C^{-|\beta|}\mathscr{E}P_n^\beta\leqslant\mathscr{E}^n P_n^\beta(x)\leqslant C^{|\beta|}\mathscr{E}P_n^\beta$$

where the constant C is that in lemma 1.

Proof. Prove this for $\beta=1$. For any $x, y\in\mathbf{T}$, there exists a $k_0=k_0(x, y)$ such that

$$|x-y-k_0/\lambda_{n-1}|\leqslant 1/\lambda_{n-1}$$

Then

$$\begin{aligned}
\frac{1}{\lambda_{n-1}} \sum_{k=0}^{\lambda_{n-1}-1} P_n\left(y+\frac{k}{\lambda_{n-1}}\right)
&= \frac{1}{\lambda_{n-1}} \sum_{k=0}^{\lambda_{n-1}-1} P_n\left(y+\frac{k_0}{\lambda_{n-1}}+\frac{k-k_0}{\lambda_{n-1}}\right) \\
&= \frac{1}{\lambda_{n-1}} \sum_{k'=0}^{\lambda_{n-1}-1} P_n\left(y+\frac{k_0}{\lambda_{n-1}}+\frac{k'}{\lambda_{n-1}}\right) \\
&\approx \frac{1}{\lambda_{n-1}} \sum_{k=0}^{\lambda_{n-1}-1} P_n\left(x+\frac{k}{\lambda_{n-1}}\right)
\end{aligned}$$

The last relation $\approx$ results from Lemma 1 and the constant involved in $\approx$ is C. ∎

What we have just proved in Lemma 2 means that $\mathscr{E}^n P_n^\beta$ is almost a constant which is the integral of P_n^β. This allows us to show the following fundamental recurrence relation, called the *fundamental inequality*.

Proposition 1. Suppose that conditions (2) are satisfied. For any $\beta \in \mathbf{R}$ the relation

$$C^{-|\beta|} \leqslant \frac{Z_{l,n}(\beta)}{Z_{l,m}(\beta)\, Z_{m,n}(\beta)} \leqslant C^{|\beta|}$$

holds for $l \leqslant m \leqslant n$.

Proof. Notice that $P_{m,n}$ is $\mathscr{F}^m$-measurable. Then we have

$$Z_{l,n}(\beta) = \mathscr{E}[\mathscr{E}^m(P_{l,m}^\beta P_{m,n}^\beta)] = \mathscr{E}[P_{m,n}^\beta \mathscr{E}^m P_{l,m}^\beta] \approx \mathscr{E} P_{m,n}^\beta \mathscr{E} P_{l,m}^\beta$$

For the last $\approx$ we heave used the remark after Lemma 2. ∎

For $\theta \in \mathbf{R}$ consider the probability measures on $\mathbf{T}$ defined by

$$\nu_{\theta,n} = \frac{P_n^\theta(t)\, dt}{Z_n(\theta)}$$

Since $\mathbf{T}$ is compact, the sequence $(\nu_{\theta,n})$ admits some weak limit points in the space of all probability measures on $\mathbf{T}$. We shall see that these limits are mutually absolutely continuous. Thus, to some extent there is a unique limit measure. For $\theta = 1$ this unique measure is just our subject of study. The family obtained while θ varies will be called the family of *Gibbs measures*.

Proposition 2. Suppose that conditions (1) and (2) are satisfied. All of the weak limits of $(\nu_{\theta,n})$ are mutually absolutely continuous. Moreover, if ν_θ denotes an arbitrary limit, we have the following approximation:

$$\nu_\gamma(I(t)) \approx |I(t)| \frac{P_n^\theta(t)}{Z_n(\theta)}$$

holds for any interval $I(t)$ containing t such that $|I(t)| \approx \lambda_n^{-1}$.

Proof. Suppose, without loss of generality, that $I(t)$ is strictly contained in an interval I' of length λ_n^{-1} [in general there is a finite number of such intervals whose union contains $I(t)$]. Then choose a continuous function h such that

$$1_{I(t)} \leqslant h \leqslant 1_{I'}$$

where $1_{I'}$ means the characteristic function of the interval I'. Now, if $\nu_\theta = \lim_j \nu_{\theta, m_j}$, we have

$$\begin{aligned}
\nu_\gamma(I(t)) &\leqslant \int_0^1 h(s)\, d\nu_\gamma(s) \\
&= \lim_{j\to\infty} \frac{1}{Z_{m_j}(\theta)} \int_0^1 h(s)\, P_{m_j}^\theta\, ds \\
&\leqslant C' \limsup_{j\to\infty} \frac{1}{Z_n(\theta)\, Z_{n,m_j}(\theta)} \int_0^1 h(s)\, P_n^\theta(s)\, P_{n,m_j}^\theta(s)\, ds \\
&\leqslant C'' \limsup_{j\to\infty} \frac{P_n^\theta(t)}{Z_n(\theta)\, Z_{n,m_j}(\theta)} \int_{I'} \prod_{k=n}^{m_j} g_k^\theta(\lambda_k s)\, ds \\
&= C'' \frac{P_n^\theta(t)}{\lambda_n Z_n(\theta)} \limsup_{j\to\infty} \frac{1}{Z_{n,m_j}(\theta)} \int_0^1 \prod_{k=n}^{m_j} g_k^\theta\left(\frac{\lambda_k}{\lambda_n} s\right) ds
\end{aligned}$$

We have used Lemma 3 for the third line and Lemma 1 for the next to the last. The last equality is obtained by changing the variable of the integral. To finish the proof of the upper estimate, we have only to show

$$Z_{n,m_j}(\theta) = \int_0^1 \prod_{k=n}^{m_j} g_k^\theta\left(\frac{\lambda_k}{\lambda_n} s\right) ds$$

This is verified by the change of variable $t = s/\lambda_n$. Observe that the function under the sign of integration is periodic because $\lambda_n \mid \lambda_k$. The lower estimate can be proved in the same way. In this case we choose an interval I' strictly contained in $I(t)$ of length $1/\lambda_{n+r}$ for some $r \geqslant 1$. We point out that the boundedness of λ_{n+1}/λ_n is then needed. ∎

The conclusion of this section is that the infinite product is well defined under conditions (1) and (2) and there is a family of Gibbs measures associated to it.

3. τ-FUNCTION, φ-FUNCTION, AND LARGE DEVIATION

Given an infinite product μ defined by $\{g_n\}$ based on $\{\lambda_n\}$, we define its *τ-function* by

$$\tau(\beta) = \lim_{r \to 0} \frac{1}{\log r} \log \int_0^1 \mu(I_r(t))^{\beta - 1} \, d\mu(t)$$

where $I_r(t)$ is the interval centered at t of length $2r$, and its *φ-function* by

$$\varphi(\beta) = \lim_{n \to \infty} \frac{\log Z_n(\beta)}{\log \lambda_n}$$

If these limits do exist for some β, we say $\tau(\beta)$ and $\varphi(\beta)$ are well defined. In general, these functions are not well defined. But they are well defined at the same time. We shall see that in some special cases (see Sections 6 and 7) they are really well defined. Here we study τ and φ and their relation to large-deviation results.

The function φ being convex, we denote its *conjugate* by

$$\varphi^\star(\alpha) = \sup\{\alpha\beta - \varphi(\beta)\}$$

The function τ being concave, we denote its *conjugate* by

$$\tau^*(\alpha) = \inf\{\alpha\beta - \tau(\beta)\}$$

The conjugates $\varphi^\star$ and τ^* are also called the Legendre transforms of φ and τ.

Let us recall some standard properties of φ and $\varphi^\star$ which will be used in the sequel. See ref. 26 for general information on convex functions and their conjugates. A real z is called a *subderivative* of φ at β if

$$\varphi(\beta + t) - \varphi(\beta) \geqslant zt \qquad (\forall t \in \mathbf{R})$$

The *subdifferential* of φ at β, denoted $\partial\varphi(\beta)$, is defined as the set of all subderivatives of φ at β. Here are some properties we shall need:

(a) $\alpha\beta \leqslant \varphi(\beta) + \varphi^\star(\alpha)$, $\forall \alpha, \beta \in \mathbf{R}$.

(b) $\alpha\beta = \varphi(\beta) + \varphi^\star(\alpha) \Leftrightarrow \alpha \in \partial\varphi(\beta)$.

(c) $\alpha \in \partial\varphi(\beta) \Leftrightarrow \beta \in \partial\varphi^\star(\alpha)$.

(d) $\varphi^{\star\star} = \varphi$.

Proposition 3. Under conditions (1) and (2), $\tau(\beta)$ and $\varphi(\beta)$ are well defined at the same time and they are related by

$$\tau(\beta) = \beta + \beta\varphi(1) - 1 - \varphi(\beta)$$

Their conjugates are related by

$$\tau^*(\alpha) + \varphi^\star(1 + \varphi(1) - \alpha) = 1$$

Proof. Let $r = 1/\lambda_n$. By Lemma 1 we have

$$\begin{aligned}
\int_0^1 \mu(I_r(t)^{\beta-1})\, d\mu(t) &= \sum_{j=0}^{\lambda_n - 1} \int_{[j/\lambda_n, (j+1)/\lambda_n]} \mu(I_r(t)^{\beta-1})\, d\mu(t) \\
&\approx \sum_{j=0}^{\lambda_n - 1} \left(\frac{1}{\lambda_n}\frac{P_n(j/\lambda_n)}{Z_n(1)}\right)^{\beta-1} \frac{1}{\lambda_n}\frac{P_n(j/\lambda_n)}{Z_n(1)} \\
&= \frac{1}{\lambda_n^{\beta-1} Z_n(1)^\beta} \sum_{j=0}^{\lambda_n - 1} \left[P_n\left(\frac{j}{\lambda_n}\right)\right]^\beta \frac{1}{\lambda_n} \\
&\approx \frac{1}{\lambda_n^{\beta-1} Z_n(1)^\beta} \int_0^1 [P_n(t)]^\beta\, dt
\end{aligned}$$

Then

$$\begin{aligned}
&\log \int_0^1 \mu(I_r(t)^\beta)\, d\mu(t) \\
&\quad = -(\beta - 1)\log\lambda_n - \beta\log Z_n(1) + \log Z_n(\beta) + O(1)
\end{aligned}$$

The relation between τ and φ follows. From this relation we can obtain that of the conjugates by using their definitions. ∎

Recall Ellis's result on large deviations.[8] Let (W_n) be a sequence of real variables defined on a probability space $(\Omega, \mathscr{A}, \sigma)$ and (a_n) be a sequence of positives. Assume that for all $\beta \in \mathbf{R}$

$$c_n(\beta) = \frac{1}{a_n}\log \mathbf{E}e^{\beta W_n}$$

is finite and the limit

$$c(\beta) = \lim_{n \to \infty} c_n(\beta)$$

exists. We call $c(\beta)$ the *free energy function* of (W_n) with respect to σ [weighted by (a_n)]. The case where $a_n = \log \lambda_n$ is the interesting case for us. The upper large-deviation bound says that under the assumption made on $c(\beta)$, for any closed set $K \subset \mathbf{R}$ we have

$$\limsup_{n \to \infty} \frac{1}{a_n} \log \sigma\{a_n^{-1} W_n \in K\} \leqslant -\inf_{\alpha \in K} c^\star(\alpha)$$

Now consider the variables defined on $\mathbf{T}$ by

$$W_n = \log P_n \qquad (n \geqslant 1)$$

We are going to study the behavior of W_n with respect to ν_θ, normalized if necessary. In order to apply the large-deviation result just mentioned we ought to study the free energy function of our sequence $W_n = \log P_n$ with respect to ν_θ, which is actually defined by

$$F_\theta(\beta) = \lim_{n \to \infty} \frac{1}{\log \lambda_n} \log \int_0^1 P_n^\beta(t)\, d\nu_\theta(t)$$

Notice that ν_0 is simply the Lebesgue measure. So $F_0(\beta) = \varphi(\beta)$, which is just the free energy function of $\{W_n\}$ with respect to the Lebesgue measure weighted by $\{\log \lambda_n\}$. A useful fact is that for each θ, F_θ is directly related to φ.

Proposition 4. Under conditions (1) and (2), we have

$$F_\gamma(\beta) = \varphi(\beta + \theta) + \varphi(\theta)$$

$$F_\theta^\star(\alpha) = \varphi^\star(\alpha) + \varphi(\theta) - \alpha\theta$$

Proof. The second equality follows from the first one. For the first, we have only to observe

$$\int_0^1 P_n^\beta\, d\nu_\theta(t) = \frac{1}{\lambda_n} \sum_I \int_I \frac{P_n^{\beta+\theta}(t)}{Z_n(\theta)}\, dt \approx \frac{Z_n(\beta+\theta)}{Z_n(\theta)}$$

where the sum is taken over intervals $I = [i/\lambda_n, (i+1)/\lambda_n]$ $(0 \leqslant i < \lambda_n)$. Lemma 1 is used for the last relation. ∎

Consequently we have

$$F_\theta^\star(\alpha)=0 \Leftrightarrow \alpha \in \partial\varphi(\theta)$$

By the positivity, convexity, and compactness of level sets of $F_\theta^\star$, we know $\partial\varphi(\theta)$ is a compact interval. Let

$$\Delta_\theta^- = \min\{x : x \in \partial\varphi(\theta)\}, \qquad \Delta_\theta^+ = \max\{x : x \in \partial\varphi(\theta)\}$$

According to the upper large-deviation bound mentioned above, for any $\delta > 0$ there is a positive $\eta > 0$ such that

$$\nu_\theta\{t \in \mathbf{T} : A_n(t) \notin [\Delta_\theta^- - \delta, \Delta_\theta^- + \delta)\} \leqslant \frac{1}{\lambda_n^\eta}$$

where $A_n(t)$ is the average

$$A_n(t) = \frac{\log P_n(t)}{\log \lambda_n}$$

Then the Borel–Cantelli lemma applies. Thus we have proved the following.

Proposition 5. Under conditions (1)–(3) we have

$$\Delta_\theta^- \leqslant \liminf_{n\to\infty} \frac{\log P_n(t)}{\log \lambda_n} \leqslant \limsup_{n\to\infty} \frac{\log P_n(t)}{\log \lambda_n} \leqslant \Delta_\theta^+$$

ν_θ-almost everywhere.

4. DIMENSIONS

First we recall the definitions of Hausdorff dimension and packing dimension and a method for calculating these dimensions of a set based on the measures supported by the set. Then we translate our τ-function as L^β-dimensions, which have recently been the subject of intensive investigation.[17-19, 30, 31]

For $0 \leqslant s < \infty$ and $A \subset \mathbf{T}$ we define for any $0 \leqslant \delta < \infty$

$$\mathscr{H}_\delta^s(A) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s$$

where the infimum is taken over all coverings $\{E_i\}$ of A by closed intervals of diameter at most δ, i.e., $A \subset \bigcup_{i=1}^{\infty} E_i$ and $\operatorname{diam} E_i \leqslant \delta$. The $\mathscr{H}_\delta^s(A)$ is obviously decreasing as a function of δ. Then we define the s-dimensional *Hausdorff measure* of A by

$$\mathscr{H}^s(A) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(A)$$

and the *Hausdorff dimension* of A as

$$\dim A = \sup\{s : \mathscr{H}^s(A) = \infty\} = \inf\{s : \mathscr{H}^s(A) = 0\}$$

$\mathscr{H}^s$ is an outer measure and Borel regular.[27] The packing dimension is defined in a slightly different way. First let

$$P^s_\delta(A) = \sup \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s$$

where the supremum is taken over all disjoint families (packings) $\{E_i\}$ of closed intervals $\{E_i\}$ such that $\operatorname{diam} E_i \leqslant \delta$ and the centers of the E_i are in A. Here $P^s(A) = \lim_{\delta \to 0} P^s_\delta(A)$ is also well defined, but is not an outer measure. Then we define the s-dimensional packing measure of A by

$$\mathscr{P}^s(A) = \inf \sum_{i=1}^{\infty} P^s(A_i)$$

where the infimum is taken over all decompositions of A, i.e., $A = \bigcup_{i=0}^{\infty} A_i$. The *packing dimension* of A, denoted Dim, is defined in the same way as the Hausdorff dimension, which means

$$\operatorname{Dim} A = \sup\{s : \mathscr{P}^s(A) = \infty\} = \inf\{s : \mathscr{P}^s(A) = 0\}$$

There is a very useful method for the calculation of Hausdorff dimension and packing dimension of a set (see, e.g., ref. 20), which we state as the following proposition. To describe this method, we introduce a notation which is also useful in the next section. For a Radon measure μ on $\mathbf{T}$ and a point $x \in \mathbf{T}$, define

$$\bar{D}(\mu, x) = \limsup_{r \to 0} \frac{\log \mu(I_r(x))}{\log r}$$

We also define $\underline{D}(\mu, x)$ in the same way by replacing lim sup by lim inf. If $\underline{D}(\mu, x) = \bar{D}(\mu, x)$, the common value is denoted by $D(\mu, x)$.

Proposition 6. Let μ be a Radon measure on $\mathbf{T}$. For any $t \geqslant 0$ we have

$$\dim\{x: \underline{D}(\mu, x) \leqslant t\} \leqslant t, \qquad \operatorname{Dim}\{x: \bar{D}(\mu, x) \leqslant t\} \leqslant t$$

If A is a set of $\mathbf{T}$ with $\mu(A) > 0$ and $\underline{D}(\mu, x) \geqslant t$ for every $x \in A$, then $\dim A \geqslant t$. Similarly, if B is a set of $\mathbf{T}$ with $\mu(B) > 0$ and $\bar{D}(\mu, x) \geqslant t$ for every $x \in B$, then $\operatorname{Dim} B \geqslant t$.

We now recall several notions of the dimension of a measure μ. In ref. 9 the upper and lower (Hausdorff) dimensions of μ are defined by

$$\dim^* \mu = \inf\{t \geqslant 0: \underline{D}(\mu, \cdot) \leqslant t \mu\text{-a.e}\}$$

$$\dim_* \mu = \sup\{t \geqslant 0: \underline{D}(\mu, \cdot) \geqslant t \mu\text{-a.e}\}$$

In a similar way, we can define the upper and lower (packing) dimensions of μ by using $\bar{D}(\mu, \cdot)$ instead of $\bar{D}(\mu, \cdot)$.[32] Denote these two dimensions by $\operatorname{Dim}^* \mu$ and $\operatorname{Dim}_* \mu$. As a consequence of the theorem in the next section, we have the following result.

Proposition 7. Suppose conditions (1)–(3) are satisfied. For the infinite product μ we have

$$1 + \varphi(1) - \Delta_1^+ \leqslant \dim_* \mu \leqslant \dim^* \mu \leqslant 1 + \varphi(1) - \Delta_1^-$$

$$1 + \varphi(1) - \Delta_1^+ \leqslant \operatorname{Dim}_* \mu \leqslant \operatorname{Dim}^* \mu \leqslant 1 + \varphi(1) - \Delta_1^-$$

If φ is differentiable at 1, all these dimensions are equal to $\tau'(1) = 1 + \varphi(1) - \varphi'(1)$.

Each of these definitions attempts to capture the idea that the dimension of μ is equal to α if $\mu(I_r(x))$ behaves like r^α almost everywhere. The L^β-dimension of μ is defined in order to capture the idea that $\mu(I_r(x))$ behaves like r^α in the sense of the L^β average.

Let $\mathscr{I}_n$ be the family of intervals $I = [i/\lambda_n, (i+1)/\lambda_n]$ $(0 \leqslant i < \lambda_n)$. For $\beta > 1$ the upper L^β dimension of μ is defined by

$$\overline{\dim}_\beta \mu = \limsup_{n \to \infty} \frac{\log \sum_{I \in \mathscr{I}_n} \mu(I)^\beta}{(\beta - 1) \log |I|}$$

and the lower L^β-dimension $\underline{\dim}_\beta \mu$ is similarly defined by the lim inf. If the two dimensions are equal, we write $\dim_\beta \mu$ for the common value, called the L^β-dimension. The definition given here is slightly different from the usual one, but for an infinite product under conditions (1) and (2) it is the same as the usual one. Actually, if condition (3) is also satisfied, we have

$$\dim_\beta \mu = \frac{\tau(\beta)}{\beta - 1}$$

5. MULTIFRACTAL ANALYSIS

Now we can prove the singularity spectrum formula.

Let μ be a measure and I be an interval. Denote

$$\underline{E}_I = \{x \in \mathbf{T}: \underline{D}(\mu, x) \in I\}$$

$$\bar{E}_I = \{x \in \mathbf{T}: \bar{D}(\mu, x) \in I\}$$

$$E_I = \{x \in \mathbf{T}: D(\mu, x) \in I\}$$

If $I = \{\alpha\}$, we shall write $\underline{E}_\alpha$, $\bar{E}_\alpha$, and E_α. The measure μ does not figure in the notation, but there is no confusion because we always talk about a fixed infinite product.

Just like $\partial\varphi(\theta), \partial\tau(\theta)$ is also an interval. We denote it by $[\theta^-, \theta^+]$. This means $\theta^- = \min \partial\tau(\theta)$ and $\theta^+ = \max \partial\tau(\theta)$. Note that the subderivatives of τ are nonnegative because τ is nondecreasing, so $\theta^+ \geqslant \theta^- \geqslant 0$.

Recall that since τ is concave, a real z is a subderivative of τ at θ [i.e., $\theta \in \partial\tau(\theta)$] if

$$\tau(\theta + t) - \tau(\theta) \leqslant zt \qquad (\forall t \in \mathbf{R})$$

As τ and φ are related by a linear function (Proposition 3), we have the following relation for their subdifferentials:

$$[\theta^-, \theta^+] = 1 + \varphi(1) - [\Delta_\theta^+, \Delta_\theta^-]$$

Define

$$\bar{A}(x) = \limsup_{n \to \infty} \frac{\log P_n(x)}{\log \lambda_n}$$

Similarly we define $\underline{A}(x)$ and $A(x)$.

Proposition 8. Suppose that conditions (1)–(3) are satisfied. We have

$$\bar{D}(\nu_\theta, x) = 1 - \theta \underline{A}(x) + \varphi(\theta)$$

Let $\alpha \in \mathbf{R}$ and $x \in \mathrm{T}$. The following facts are equivalent:

1. $\bar{D}(\mu, x) = \alpha$.
2. $\underline{A}(x) = 1 + \varphi(1) - \alpha$.
3. $\bar{D}(\nu_\theta, x) = \theta\alpha - \tau(\theta)$.

The same relations hold for $\underline{D}$ and D with $\underline{A}$ replaced, respectively, by $\bar{A}$ and A.

Proof. Let $r \approx 1/\lambda_n$. By Proposition 1 we have

$$\nu_\theta(I_r(x)) \approx \frac{1}{\lambda_n} \frac{P_n^\theta(x)}{Z_n(\theta)}$$

from which the first equality follows immediately. Let $\theta = 1$. We have, in particular, the equality

$$\bar{D}(\mu, x) = 1 - \underline{A}(x) + \varphi(1)$$

which implies immediately the equivalence of facts 1 and 2. If $\bar{D}(\mu, x) = \alpha$, substituting $\underline{A}(x) = 1 + \varphi(1) - \alpha$ into the equality proved above gives

$$\begin{aligned} \bar{D}(\nu_\theta, x) &= 1 - \theta[1 + \varphi(1) - \alpha] + \varphi(\theta) \\ &= \theta\alpha - [\theta - 1 + \theta\varphi(1) - \varphi(\theta)] \end{aligned}$$

Thus we obtain the equivalence of facts 2 and 3 if we use the relation between φ and τ established in Proposition 3. ∎

As a direct consequence of the preceding proposition and Proposition 5, we have the following result.

Proposition 9. Under conditions (1)–(3) we have

$$1 - \theta\Delta_\theta^+ + \varphi(\theta) \leqslant \underline{D}(\nu_\theta, x) \leqslant \bar{D}(\nu_\theta, x) \leqslant 1 - \theta\Delta_\theta^- + \varphi(\theta)$$

ν_θ-almost everywhere.

Theorem 1. Suppose that conditions (1)–(3) are satisfied.

1. For $\theta \geqslant 0$ we have

$$\tau^*(\theta^-) \leqslant \dim \underline{E}_{[\theta^-, \theta^+]} \leqslant \tau^*(\theta^+)$$
$$\tau^*(\theta^-) \leqslant \mathrm{Dim}\, \bar{E}_{[\theta^-, \theta^+]} \leqslant \tau^*(\theta^+)$$

For $\theta \leqslant 0$ the above inequalities hold if we change the roles of $\tau^*(\theta^-)$ and $\tau^*(\theta^+)$.

2. If $\alpha=\tau'(\theta)$ for some θ we have

$$\dim E_\alpha=\dim \underline{E}_\alpha=\dim \bar{E}_\alpha=\operatorname{Dim} E_\alpha=\operatorname{Dim} \underline{E}_\alpha=\operatorname{Dim} \bar{E}_\alpha=\tau^*(\alpha)$$

Proof. 1. Suppose $\theta\geqslant 0$. Notice that

$$\underline{E}_{[\theta^-,\theta^+]}=\{x: \theta\theta^- - \tau(\theta)\leqslant \underline{D}(v_\theta,x)\leqslant \theta\theta^+ - \tau(\theta)\}$$

Notice also that

$$\theta\theta^- - \tau(\theta)=1-\theta\Delta_\theta^+ + \varphi(\theta)$$
$$\theta\theta^+ - \tau(\theta)=1-\theta\Delta_\theta^- + \varphi(\theta)$$

According to the last proposition, there exists a Borel set F_θ such that $v_\theta(F_\theta)>0$ and for every $x\in F_\theta$

$$1-\theta\Delta_\theta^+ + \varphi(\theta)\leqslant \underline{D}(v_\theta,x)\leqslant \bar{D}(v_\theta,x)\leqslant 1-\theta\Delta_\theta^- + \varphi(\theta)$$

So $F_\theta\subset \underline{E}_{[\theta^-,\theta^+]}$. This implies, by Proposition 6, that

$$\tau^*(\theta^-)=\theta\theta^- - \tau(\theta)\leqslant \dim F_\theta\leqslant \dim \underline{E}_{[\theta^-,\theta^+]}\leqslant \theta\theta^+ - \tau(\theta)=\tau^*(\theta^-)$$

If $\theta\leqslant 0$, it suffices to note that we get inverse inqualities in the expression of $E_{[\theta^-,\theta^+]}$. The proof of the two other inequalities is exactly the same, just replacing dim by Dim and $\underline{D}$ by $\bar{D}$.

Result 2 has the same proof as result 1, but using also $D(v_\theta,x)$. ∎

6. EXISTENCE OF φ-FUNCTION: DETERMINIST CASES

We consider here a special case where $\lambda_n=q^n$ for some integer q. We prove that if $\{g_n\}$ is almost periodic in a sense we shall make precise, the corresponding φ-function is well defined. We observe also that this function is analytic if $\{g_n\}$ is periodic or eventually constant. We begin with the latter, simple case.

Theorem 2. Let $q\geqslant 2$ be an integer and $\{g_n\}$ be a strictly positive function defined on T with the property that $\Omega(\log g_n,\cdot)\leqslant\Omega(\cdot)$ for some function Ω satisfying

$$\int_0^1 \frac{\Omega(r)}{r}\,dr<\infty$$

Then the φ-function is well defined and analytic if one of the following two conditions is satisfied:

1. $\{g_n\}$ is periodic, i.e., $g_{n+p} = g_n$ for some $p \geqslant 1$.
2. There is some function g such that

$$\lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\|\log g_n - \log g\|_{C(\mathbf{T})}) = 0$$

Proof. If $\{g_n\}$ is a constant sequence, i.e., $p = 1$, the result is well known.[28] We are going to see that the results for the two cases listed above are consequences of this well-known result.

For result 1, let $\tilde{q} = q^p$ $(p \geqslant 2)$ and $\tilde{g}(x) = \prod_{j=0}^{p-1} g(q^j x)$. We have only to observe the relation

$$Z_{np}(\beta) = \prod_{k=0}^{n} \prod_{0}^{p-1} g(q^{kp+j}x) = \prod_{k=0}^{n} \tilde{g}(\tilde{q}^k x)$$

For result 2, we observe first that $|\log g(x) - \log g(y)|$ is bounded by

$$|\log g(x) - \log g_n(x)| + |\log g_n(x) - \log g_n(y)| + |\log g_n(y) - \log g(y)|$$

So, if $|x - y| < \delta$, for any $\varepsilon > 0$, for large $n \geqslant n_\varepsilon$ we have

$$|\log g(x) - \log g(y)| \leqslant 2\varepsilon + \Omega(\delta)$$

Then $\Omega(\log g, \cdot) \leqslant \Omega(\cdot)$. Observe also that the condition implies that

$$\log P_n(x) = \log \prod_{j=0}^{n-1} g(q^j x) + o(n)$$

holds uniformly in x. Thus the φ-function is the same as that defined by g. ∎

Given a sequence of functions $\{g_n\}$ in $C(\mathbf{T})$, it is said to be *almost periodic* if for any $\varepsilon > 0$ there exists a positive $l > 0$ such that any interval of length l contains an integer τ such that

$$\|g_{n+\tau} - g_n\|_{C(\mathbf{T})} \leqslant \varepsilon \qquad (\forall n)$$

For example, let $h \in C(\mathbf{T})$ and let $\{a_n\}$ be a sequence of real number which is almost periodic in Bohr sense and satisfies $\sup_{n \geqslant 0} \|a_n\| < 1/\|h\|_{C(\mathbf{T})}$. Construct two sequences of functions $\{f_n\}$ and $\{g_n\}$ by $f_n = 1 + a_n h$ and $g_n = \log f_n$. It is easy to verify that these two sequences are almost periodic.

Theorem 3. Let $\{\lambda_n\} = q^n$ for some $q \geqslant 2$, and let $\{g_n\} \subset C(\mathbf{T})$ be a sequence with the properties that $\{\log g_n\}$ is almost periodic and that $\|g_n\| \approx \|g_{n+1}\|$. If condition (2) is satisfied, then for every $\beta \in \mathbf{R}$, $\varphi(\beta)$ is well defined.

Proof. First we make a remark which generalizes a known fact about the convergence of subadditive sequences. Let $\{a_n\}$ be a sequence of real numbers. Suppose for any $\varepsilon > 0$ there exist two constants $A = A_\varepsilon$ and $B = B_\varepsilon$ such that

$$a_{n+m} \leqslant a_n + a_m + A + \varepsilon Bm \qquad (\forall n, m)$$

Then the sequence $\{a_n/n\}$ converges. In fact, for any $k \geqslant 1$, by induction we have

$$a_{km} \leqslant k(a_m + A + \varepsilon Bm) \qquad (\forall m)$$

Write $n = km + i$ with $0 \leqslant i < m$; we have

$$a_n \leqslant k(a_m + A + \varepsilon Bm) + (\max_{0 \leqslant i < m} a_i + A + \varepsilon Bm) \qquad (\forall m)$$

Divide this inequality by n and then let $n \to \infty$. We have

$$\limsup \frac{a_n}{n} \leqslant \frac{a_m}{m} + \frac{A}{m} + \varepsilon B$$

It follows that $\limsup a_n/n \leqslant \liminf a_m/m$.

Let $\varepsilon > 0$. Since $\{\log g_n\}$ is almost periodic, there exists l with the property that for any n we can find τ such that $|n - \tau| \leqslant l/2$ and

$$|\log g_{\tau+k}(x) - \log g_k(x)| \leqslant \varepsilon \qquad (\forall x \in \mathbf{T}, \forall k)$$

So, if $M = \sup_n(\|g_n\|/\|g_{n+1}\| + \|g_{n+1}\|/\|g_n\|)$,

$$\begin{aligned} Z_{n,n+m}(\beta) &= \int_0^1 \prod_{k=0}^{m-1} g_{n+k}^\beta(q^k x) \\ &\leqslant M^{l|\beta|} \int_0^1 \prod_{k=0}^{m-1} g_{\tau+k}^\beta(q^k x) \\ &\leqslant M^{l|\beta|} e^{\varepsilon m} \int_0^1 \prod_{k=0}^{m-1} g_k^\beta(q^k x) \end{aligned}$$

Since the last integral is $Z_m(\beta)$, by using Proposition 1 we have

$$\log Z_{n+m}(\beta) \leqslant \log Z_n(\beta) + \log Z_m(\beta) + |\beta| \log C + l\,|\beta| \log M + \varepsilon m$$

Finally we have only to apply the remark. ▌

7. EXISTENCE OF φ-FUNCTION: RANDOM CASES

Let X be a probability space and let $\Psi: X \to C(\mathbf{T})$ be a measurable mapping. We write $\Psi(\xi) = g_\xi(\cdot)$. Let $\omega = (\omega_n)$ be a sequence with elements in X. We are going to choose our generating functions according to ω as follows:

$$g_n^{(\omega)} = \Psi(\omega_n) = g_{\omega_n}$$

The corresponding functions will be denoted respectively by $P_n^{(\omega)}$, $Z_n^{(\omega)}$, $\varphi^{(\omega)}$, etc.

Theorem 4. Suppose there exists a function $\Omega(\cdot)$ such that $\Omega(f, \cdot) \leqslant \Omega(\cdot)$ for every function f in the image of Ψ and

$$\int_0^1 \frac{\Omega(r)}{r}\, dr < \infty$$

If $\omega = (\omega_n)$ is stationary, for almost surely all ω the following limit exists for every $\beta \in \mathbf{R}$:

$$\varphi^{(\omega)}(\beta) = \lim_{n \to \infty} \frac{1}{n} \log_q Z_n^{(\omega)}(\beta)$$

Furthermore, if this sequence is ergodic, the limit is independent of ω and is equal to

$$\varphi(\beta) = \lim_{n \to \infty} \frac{1}{n} \mathbf{E} \log_q Z_n^{(\omega)}(\beta)$$

Moreover,

$$\left| \varphi(\beta) - \frac{1}{n} \mathbf{E} \log_q Z_n^{(\omega)}(\beta) \right| \leqslant \frac{|\beta| \log C}{n}$$

Proof. For fixed β let $f_n(\omega) = \log_q Z^{(\omega)}(\beta)$. According to Proposition 1, we have

$$f_{n+m}(\omega) = f_n(\omega) + f_m(T^n \omega) + O(1)$$

for $n, m \geqslant 1$, where $O(1)$ is bounded by $|\beta| \log C$. As (ω_n) is stationary, the Kingman subadditive ergodic theorem implies the existence of the limit in question and confirms that the limit is $\varphi(\beta)$ in the case of ergodicity.[16] Consequently, for almost all ω the limit exists for a *countably* dense set of $\beta \in \mathbf{R}$. Now, observe the convexity of $\log Z_n^{(\omega)}(\beta)$ as function of β. According to a convergence property of sequences of convex functions (ref. 26, p. 90), for almost all ω the limit exists for *all* $\beta \in \mathbf{R}$ and is even uniform on each compact subset of $\mathbf{R}$.

The last equality allows us to write

$$f_{nm}(\omega) = \sum_{k=0}^{m-1} f_n(T^{nk}\omega) + O(m)$$

Taking expectations and dividing by nm, we have then

$$\frac{1}{nm} \mathbf{E} f_{nm} = \frac{1}{n} \mathbf{E} f_n + O\left(\frac{1}{n}\right)$$

Here we have used the stationarity of (ω_n). Finally, let $m \to \infty$. ∎

Proposition 10. Suppose the hypotheses in the last theorem are satisfied. Suppose, moreover, that $\{\omega_n\}$ is an i.i.d. sequence with common distribution F. Then we have

$$\varphi(\beta) \leqslant \log_q \iint g_{\cdot}^{\beta}(t)\, dt\, dF(\cdot)$$

We just have to apply the Hölder inequality to the expression for φ in the last theorem. In ref. 11 it was announced that the equality is valid, but we had some difficulty proving it. However, the following remark might be useful in an attempt to prove the equality.

Fix β. Let

$$Y_n(\omega) = \frac{1}{n} \log_q Z_n^{(\omega)}(\beta)$$

By Proposition 1,

$$-|\beta| \log_q C \leqslant \log_q Z_{n+m}^{(\omega)}(\beta) - [\log_q Z_n^{(\omega)}(\beta) + \log_q Z_m^{(\omega)}(\beta)] \leqslant |\beta| \log_q C$$

If follows that

$$\left| Y_{nm}(\omega) - \frac{1}{m} \sum_{j=0}^{m-1} Y_n(T^{jn}\omega) \right| \leqslant \frac{|\beta| \log_q C}{n}$$

Consider the Laplace transform of Y_n defined by

$$L_n(\xi) = \mathbf{E} \exp \xi Y_n \qquad (\xi \in \mathbf{R})$$

Notice that the terms in the last sum are *independent*. We can obtain

$$e^{-(|\xi\beta| \log_q C)/n} [L_n(\xi/m)]^m \leqslant L_{nm}(\xi) \leqslant e^{(|\xi\beta| \log_q C)/n} [L_n(\xi/m)]^m$$

8. EXAMPLES

Let us consider some examples.

The first example is the Riesz product

$$\mu = \prod_{n=0}^{\infty} (1 + \operatorname{Re} a_n e^{2\pi i \lambda_n t})$$

where (a_n) is a sequence of complex numbers with $|a_n| \leqslant 1$. The product makes sense, for it does converge weakly,[35] even if the corresponding generating functions take zero as a value, i.e., if $|a_n| = 1$. In the sequel, we assume $|a_n| \leqslant r < 1 (\forall n)$ for some r. For such products, multifractal analysis has been done in the case where the sequence (a_n) is constant and $\lambda_n = q^n$ for some integer q.[6,9] As consequences of the results we have obtained in the preceding sections, we can add the following cases: Recall that we suppose $\lambda_n = q^n$,

(i) (a_n) is periodic or almost periodic.

(ii) (a_n) is such that there is a real number c such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |a_n - c| = 0$$

(iii) Almost surely every sequence of an ergodic sequence $(a_n(\omega))$.

A special case of (i) is $a_n = re^{in\alpha}$ for some $0 < r < 1$ and some real number α. If α is rational, (a_n) is periodic; if α is irrational, (a_n) is almost periodic. Notice that there is a general way in ergodic theory to produce almost-periodic sequences from ergodic isometry.[23] The conclusion of (ii) is that the multifractal property of the Riesz product μ_a defined by (a_n) is the same as the Riesz product defined by the constant sequence taking value c. Thus, if we have another Riesz product μ_b defined by a sequence (b_n) such that $n^{-1} \sum_{j=0}^{n-1} |a_n - b_n|$ tends to zero, then μ_a and μ_b share the same multifractal property. We should point out that there are couples of μ_a, μ_b which are singular with respect to each other, e.g., $a_n = r$ and $b_n = r + n^{-\varepsilon}$ for some $0 < \varepsilon \leqslant 1/2$.[24]

These Riesz products are merely special cases of G-measures,[3] which include g-measures.[15] Let us recall the definition of G-measures, which are defined as ergodic measures under the action of the direct sum of finite groups. Let $\{X_k\}(k \geqslant 1)$ be a sequence of finite Abelian groups of orders $\{m_k\}$. Let X be the infinite product and Γ be the infinite direct sum of groups in this sequence. Γ can be identified with a subgroup of X and then it acts on X. For $\gamma = (\gamma_n) \in \Gamma$ (there are finitely many nonnull coordinates of γ) and $x = (x_n) \in X$, the action of γ on x is denoted by γx, i.e., $\gamma x = (\gamma_n + x_n)$. Let Γ_n be the direct sum of $X_k (1 \leqslant k \leqslant n)$. We shall be interested in the actions of these subgroups of Γ. The product topology makes X a compact metrizable space. Suppose we are given a sequence of continuous nonnegative functions $\{g_n\}$, i.e., $0 \leqslant g_n \in C(X)$, such that g_n is Γ_{n-1}-*invariant* [i.e., $g_n(x) = g_n(\gamma x)$ for all $\gamma \in \Gamma_{n-1}$ and all $x \in X$] and *normalized* in that

$$\frac{1}{m_n} \sum_{\gamma \in \Gamma_n} g_n(\gamma x) = 1 \qquad (\forall x \in X)$$

A measure is called a G-measure if $d\mu/d\mu_n = G_n(x)$, where

$$G_n(x) = \prod_{k=1}^{n} g_k(x)$$

$$\mu_n = \frac{1}{m_1 m_2 \cdots m_n} \sum_{\gamma \in \Gamma_n} \mu \circ \gamma$$

($\mu \circ \gamma$ is the image of μ under the action of γ). Here G means the sequence $G = \{G_n\}$. The uniqueness of such G-measures is studied in refs. 3 and 10, where some sufficient conditions are found. For example, if the g_n are strictly positive and if the functions $\log g_n$ satisfy a Lipschitz condition, then there is a unique G-measure.

Our study of infinite products on $\mathbf{T}$ can be translated word for word into the situation for G-measures on X. Actually, in the case where $X_k = \mathbf{Z}(m_k)$, there is a canonical mapping from X onto $\mathbf{T}$ defined by

$$\Pi(x) = \sum_{k=1}^{\infty} \frac{x_k}{m_1 \cdots m_k}$$

We do not restate the results for G-measures because they are exactly the same. We should point out that it is the method which can be adapted in the case of G-measures, not the results which can be applied to G-measures. We should have started with G-measures. Our choice was made because of the concreteness of infinite products. Also, we should point out that even for G-measures, taking powers g_n^{θ} to construct the corresponding Gibbs measures will destroy the condition of normalization. It is thus not practical to use G-measures to construct Gibbs measures. Anyway, it is not important for us whether a Gibbs measure is a G-measure or not.

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